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TECHNICAL NOTE 29-TDL-3

Atmospheric Effects on Re-Entry Vehicle Dispersions

BASIC KNOWLEDGE

TECHNIQUES DEVELOPMENT LABORATORY

OPERATING PROCEDURES

TECHNIQUES DEVELOPMENT LABORATORY REPORT NO.3

WASHINGTON,D.C.
December 1965

WEATHER BUREAU TECHNICAL NOTES

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Karl R. Johannessen

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ATMOSPHERIC EFFECTS ON RE-ENTRY VEHICLE DISPERSIONS

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ABSTRACT

Wind and density variability cause dispersion of the trajectory of re-entry vehicles and of the altitude signalled by inertial altimetry systems. These effects are analyzed and formulas are developed for them which make use of the basic trajectory data (no wind, standard density) and conventional statistical data available from the meteorological archives.

The procedure makes it possible to compute the distribution of meteorologically-induced system errors with little effort and without recourse to high-speed computers. The procedure has all the accuracy warranted by the meteorological data. It lends itself to optimization of inertial altimetry programs.

INTRODUCTION

When computing the re-entry trajectory of a vehicle, usually no wind is assumed and the density profile which forms the basis for the calculation is some standard profile representing some average conditions in the impact area.

Winds and non-standard density will cause perturbations on the standard re-entry trajectory and cause an impact error and error in the altimetry performance.

The atmosphere is constantly in motion and changes its density distribution from day-to-day in a complex manner. We are, naturally, interested in the extent to which the re-entry trajectories become affected.

The atmospheric effects, since they are uncontrollable, represent in a way the ultimate obtainable in impact accuracy. There is little point in reducing other error sources below the values of the atmospheric error sources. In this respect, the atmospheric variability of the impact region sets design standards. By translating the atmospheric variability into a corresponding distribution of impact errors or altimetry errors or whichever kind of error we happen to be interested in, the point of diminishing returns in the precision of system components may be determined.

Particularly interesting in this respect are inertially programed altimetry systems. The object of an altimetry system is to initiate an event at a selected point on the trajectory. Since inertial programs make no reference to the earth below, all information obtained by the system originates in the atmosphere and as a result, inertial altimetry systems are victims of atmospheric variability, essentially the variability of the density profile.

In an inertial altimetry system, a great variety of inertial signals may be used, both discrete deceleration events and outputs from integrating decelerometers. One may ask the question whether the character of the atmospheric variability favors certain programs. Or, in other words, do the migratory atmospheric disturbances, such as they appear in statistical summaries, define optimum altimetry programs?

The answer is, "Yes", as a rule they do. An analysis of the atmospheric effects on a proposed altimetry system becomes a prerequisite to designing one. Technology of altimetry components of the inertial type is advancing to the point where atmospheric variability is the chief residue of the altimetry error. At this stage, it may be more important to design the system in accordance with the atmospheric variability than to improve the components further.

All this points to the need for mathematical models which describe the atmospheric effects. Several approaches are possible and have been used.

The first method which suggests itself may be referred to as the Monte Carlo Method. A computer program which accepts wind and an arbitrary density profile is written for the specific re-entry problem and the results of the integration are compared with the standard trajectory solution. Deviations in the desired parameters from standard are noted and associated with the applied atmospheric conditions. The Monte Carlo Method is expensive, since it involves a complete re-entry trajectory integration for each atmospheric profile. A voluminous computer task is involved to obtain a representative statistical distribution for various seasons and localities. The job is further augmented manyfold in optimization problems where parameters of re-entry or altimetry conditions have to be varied in a multi-sided matrix, as the entire climatological sample of profiles has to be run through over again for every point in the matrix. To sum up: the Method has all the labor and lack of sophistication of brute force techniques. None of the vast amounts of statistical data on atmospheric variability which are available in the meteorological archives can be applied. While the Monte Carlo Method has merit as a check on other methods, it is not recommended for obtaining representative statistical samples.

A second method which has been used is the Method of Influence Coefficients. The re-entry problem is integrated numerically, applying either a "unit" density anomaly or a "unit" wind in a layer of the atmosphere, the rest of the profile being standard. The noted departure from standard in the result is written as a linear product of the influence coefficient and the density anomaly or the wind. Repeating for each layer in the atmosphere we assemble linear expressions of the type

$$\sum a_i \rho_i \quad \text{and} \quad \sum b_i w_i$$

where a_i and b_i are the influence coefficients for the density anomaly ρ_i and for the wind component w_i in the i-th layer, respectively. The coefficients are "empirical" in the sense that it is difficult to give them any physical interpretation and hence "understand" the meteorological effects.

A priori there is no justification for assuming a linear expression for the atmospheric effects but a physical-mathematical analysis of the trajectory perturbations caused by meteorological disturbances shows that the meteorological effects are essentially linear.

The great strength of the Method of Influence Coefficients over the Monte Carlo Method is the fact that it can make use of readily-available statistics of density and winds. A weakness of the method is the fact that the coefficients have been given no physical meaning. Their behavior for changing vehicle, re-entry, or altimetry conditions cannot be predicted. They must be derived anew each time a change in these conditions occurs. This may not be a serious objection in an era of high-speed computers.

The above-mentioned physical-mathematical analysis of the meteorological effects on re-entry leads us directly into a third method, which forms the subject of this paper. It is referred to as the Perturbation Method. The meteorological effect is obtained as a linearized perturbation on the standard trajectory.

THE PERTURBATION VELOCITIES

By comparing the vehicle velocity in the real atmosphere with the velocity in the standard atmosphere, we may define a perturbation velocity as the difference between the two. The comparison may be effected at the same points in time, or at the same points in geometric altitude, or at the same points in pressure. It turns out that the comparison of velocities at the same pressure in the real atmosphere and in the model atmosphere has particular advantages, and as a result, this is the method which will be used in the following. The three different methods of comparison lead to different definitions of the perturbation velocity and also, to different values.

Since we shall consider only the first-order part of the perturbation, we may investigate the perturbation which is caused by the wind independently of the perturbation caused by a non-standard density structure.

The wind perturbation is considered first. In an atmosphere with wind, but with standard density, the equation governing re-entry may be written as:

$$\frac{d}{dt}\left(\mathbb{C}+\mathbb{c}_v\right) = \mathbf{g} - 2\,\boldsymbol{\Omega}\times\left(\mathbb{C}+\mathbb{c}_v\right) - \frac{1}{2}\,\rho\,\frac{C_D A}{m}\left|\mathbb{C}+\mathbb{c}_v-\mathbf{v}\right|\left(\mathbb{C}+\mathbb{c}_v-\mathbf{v}\right) \tag{1}$$

The symbols are explained in the list at the end of the paper.

It is noted that it has been assumed that the re-entry vehicle at all times is oriented along the vector of the relative motion of the air with respect to the vehicle.

It is advantageous at this stage to introduce atmospheric pressure as an independent variable indicating position in the vertical since most of the atmospheric variables such as density, temperature, and wind are observed and stored as a function of pressure. The equations also assume a particularly simple form when pressure is used rather than altitude or distance along the trajectory.

The geometric relationship:

$$\sin \psi \, ds = -dz \tag{2}$$

and the hydrostatic relationship

$$dp = -g \, P \, dz \tag{3}$$

may be used to derive the relation

$$\frac{d}{dt} = \frac{ds}{dt}\frac{d}{ds} = g \, C P \sin \psi \frac{d}{dp} \tag{4}$$

By means of (4) we may rewrite equation (1):

$$g \, C P \sin \psi \frac{d}{dp}(\mathbb{C}+\mathbb{c}_v) = \mathfrak{g} - 2\,\Omega \times (\mathbb{C}+\mathbb{c}_v) - \frac{1}{2} P \frac{C_D A}{m} |\mathbb{C}+\mathbb{c}_v - \mathbb{v}| (\mathbb{C}+\mathbb{c}_v - \mathbb{v}) \tag{5}$$

In the standard atmosphere the corresponding equation is:

$$g \, C P \sin \psi \frac{d\mathbb{C}}{dp} = \mathfrak{g} - 2\,\Omega \times \mathbb{C} - \frac{1}{2} P \frac{C_D A}{m} C \mathbb{C} \tag{6}$$

Subtracting (6) from (5) gives the equation which determines the perturbation velocity $\mathbb{c}_v$ as determined by the wind $\mathbb{v}$:

$$g \, C P \sin \psi \frac{d\mathbb{c}_v}{dp} = -2\,\Omega \times \mathbb{c}_v - \frac{1}{2} P \frac{C_D A}{m} \left[|\mathbb{C}+\mathbb{c}_v - \mathbb{v}| (\mathbb{C}+\mathbb{c}_v - \mathbb{v}) - C\mathbb{C} \right] \tag{7}$$

The Coriolis term $-2\Omega \times \mathbb{c}_v$ is extremely small and may be discarded forthwith. As an example, if the total displacement due to $\mathbb{c}_v$ during a 60-sec. re-entry amounts to 1000 ft., the maximal displacement due to the Coriolis term is 4 ft. and is less than the displacement due to uncertainities of the wind measurements.

We will make use of the fact that $\mathbb{c}_v$ and $\mathbb{v}$ are small compared to $\mathbb{C}$ (of order 10^{-3} to 10^{-2}) and will be satisfied to compute first-order effects.

The second right-hand term in (7) is expanded in a power series of $\mathfrak{C}_v - \mathfrak{v})/C$ and only first-order terms are retained.

Equation (7) is now reduced to:

$$\frac{d\mathfrak{C}_v}{dp} = -\frac{C_D A}{2mg\sin\psi}\left(1+\mathfrak{T}\mathfrak{T}\cdot\right)\left(\mathfrak{C}_v-\mathfrak{v}\right) \tag{8}$$

The trajectory angle ψ as a function of pressure p may be regarded as undisturbed. Taking into account the perturbation of ψ will only introduce second-order corrections.

We now introduce a new non-dimensional pressure variable, ω, defined by

$$d\omega = \frac{C_D A}{2mg\sin\psi}\,dp \tag{9}$$

For reasonably straight trajectories where the velocity does not approach Mach 1, $C_D A/2mg\sin\psi$ may be regarded as a constant and we have:

$$\omega = \frac{C_D A}{2mg\sin\psi}\,p \tag{10}$$

As may be shown, $mg\sin\psi/C_D A$ represents the atmospheric pressure at maximum deceleration and it is seen that in the case of equation (10) ω measures atmospheric pressure in units of twice the pressure at maximum deceleration [3].

With this new pressure variable equation (8) takes on the form:

$$\frac{d\mathfrak{C}_v}{d\omega} = -\mathfrak{A}\cdot\left(\mathfrak{C}_v-\mathfrak{v}\right) \tag{11}$$

where we have used the dyadic operator

$$\mathfrak{A}\cdot \equiv \left(1+\mathfrak{T}\mathfrak{T}\cdot\right)$$

When the standard trajectory is known ω can be computed as a function of p according to (9) or (10); $\mathfrak{A}$ is also known and the perturbation velocity $\mathfrak{C}_v$ may now be computed by means of (11).

Next, we will turn to the perturbation velocity due to density anomalies.

The advantage of using pressure p or the derived quantity ω as coordinate for vertical position now becomes apparent. It can be shown that the perturbation velocity due to density, when taken as a function of pressure, is vanishingly small (of second order). In other words, the vehicle velocity as a function of pressure is independent of the density distribution. If viewed as a function of geometric altitude, or as a function of time, this is no longer the case, as the pressure surfaces move up and down.

To demonstrate the above statement, consider a wind-less atmosphere with an arbitrary density profile given by the density anomaly ρ superimposed on the standard density P. The governing equation is:

$$\frac{d}{dt}(\mathbb{C}+\mathbb{c}_\rho) = \mathbf{g} - 2\,\Omega \times (\mathbb{C}+\mathbb{c}_\rho) - \frac{1}{2}(P+\rho)\frac{C_D A}{m}|\mathbb{C}+\mathbb{c}_\rho|(\mathbb{C}+\mathbb{c}_\rho) \qquad (12)$$

Scalar multiplication by $\mathbb{C}+\mathbb{c}_\rho$ gives

$$\frac{d}{dt}\left[\frac{1}{2}(\mathbb{C}+\mathbb{c}_\rho)^2 + gz\right] = -\frac{1}{2}(P+\rho)\frac{C_D A}{m}|\mathbb{C}+\mathbb{c}_\rho|(\mathbb{C}+\mathbb{c}_\rho)^2 \qquad (13)$$

By means of the relationship expressed in (4)

$$\frac{d}{dt} = g|\mathbb{C}+\mathbb{c}_\rho|(P+\rho)\sin\psi\frac{d}{dp}$$

and by using (9), equation (13) becomes:

$$\frac{d}{dw}\left[\frac{1}{2}(\mathbb{C}+\mathbb{c}_\rho)^2 + gz\right] = -(\mathbb{C}+\mathbb{c}_\rho)^2 \qquad (14)$$

In the equation above, z is the height of the pressure surface p

Similarly, we can write for re-entry in the standard atmosphere:

$$\frac{d}{dw}\left[\frac{1}{2}\mathbb{C}^2 + gz_s\right] = -\mathbb{C}^2 \qquad (15)$$

where z_s is the height of the pressure p in the standard atmosphere.

Subtracting (15) from (14) and retaining first-order quantities, we obtain:

$$\frac{d}{dw}(\mathbb{C}\cdot\mathbb{c}_\rho + gD) = -2\,\mathbb{C}\cdot\mathbb{c}_\rho \qquad (16)$$

where D is the D-value at pressure p defined as

$$D = z - z_s \qquad (17)$$

i.e., the altitude of pressure p in the real atmosphere minus the altitude of the same pressure in the model atmosphere.

Equation (16) has the solution:

$$\mathbb{C}\cdot\mathbb{c}_{\varsigma} = -g\,D + 2g\int_0^{\omega} D e^{-2(\omega-\xi)}\,d\xi \tag{18}$$

This solution is compatible with the boundary conditions $\mathbb{c}_{\varsigma} = 0$ and $D=0$ at $\omega = 0$.

It may be shown that $\mathbb{c}_{\varsigma}$ as defined is directed along the vertical, hence, we can write (c_{ς} positive when directed upwards),

$$\mathbb{C}\cdot\mathbb{c}_{\varsigma} = -\sin\psi\, C c_{\varsigma} \tag{19}$$

and we obtain

$$c_{\varsigma} = \frac{g}{\sin\psi C}\left[D - 2\int_0^{\omega} D e^{-2(\omega-\xi)}\,d\xi\right] \tag{20}$$

$$c_{\varsigma} = \frac{g}{\sin\psi C}\left[D - (1-e^{-2\omega})\,\overline{D}\right] \tag{21}$$

The expression inside the brackets in (21) seldom exceeds 500 ft. over any interval of the re-entry. Using a value ψ = 20 deg., C = 10,000 ft. sec.$^{-1}$ we find that c_{ς} rarely exceeds 5 ft. sec.$^{-1}$ and as a result, we find that the vertical displacement due to $\mathbb{c}_{\varsigma}$ is contained within a few tens of feet of the position obtained by assuming that $\mathbb{c}_{\varsigma} = 0$. Since because of the limitations of the meteorological measuring techniques, the height of the pressure surfaces cannot be calculated with a corresponding precision, the refinement of calculating $\mathbb{c}_{\varsigma}$ becomes pointless, and we may accept the thesis proposed earlier. This states that the vehicle velocity, as a function of atmospheric pressure, is independent of the density profile. In general, we can write

$$c_{\varsigma}(p) = 0 \quad ; \quad \mathbb{C} = \mathbb{C}(p) \tag{22}$$

$$\mathbb{c}_{\varsigma}(z) = g\,P\,\frac{d\mathbb{C}}{dp}\,D \tag{23}$$

The formula for the velocity given by Allen and Eggers [1]

$$C = C_E\, e^{-\omega} \tag{24}$$

is obtained from the dynamic equation by ignoring the gravity term in comparison with the drag deceleration. This expression for the velocity makes no assumption about the density profile. It has been demonstrated above that this result is valid more generally, so that within altitude tolerances of a few tens of feet the vehicle velocity, as far as the atmosphere is concerned, is a function of pressure only and is independent of the density profile.

An example will show the magnitude of $C_S(z)$. Assuming z = 30,000 ft., D = 1,000 ft., C_E = 22,000 ft. sec.$^{-1}$, mg/C_DA = 1,500 lb. (sq.-ft)$^{-1}$, ψ = 20 deg., we find C_S (30,000 ft.) = 250 ft. sec.$^{-1}$.

Figure 1 is qualitative diagram showing how the vehicle velocity is constant at a specific pressure but decreases or increases at a specific altitude as the atmosphere becomes warmer or colder than standard.

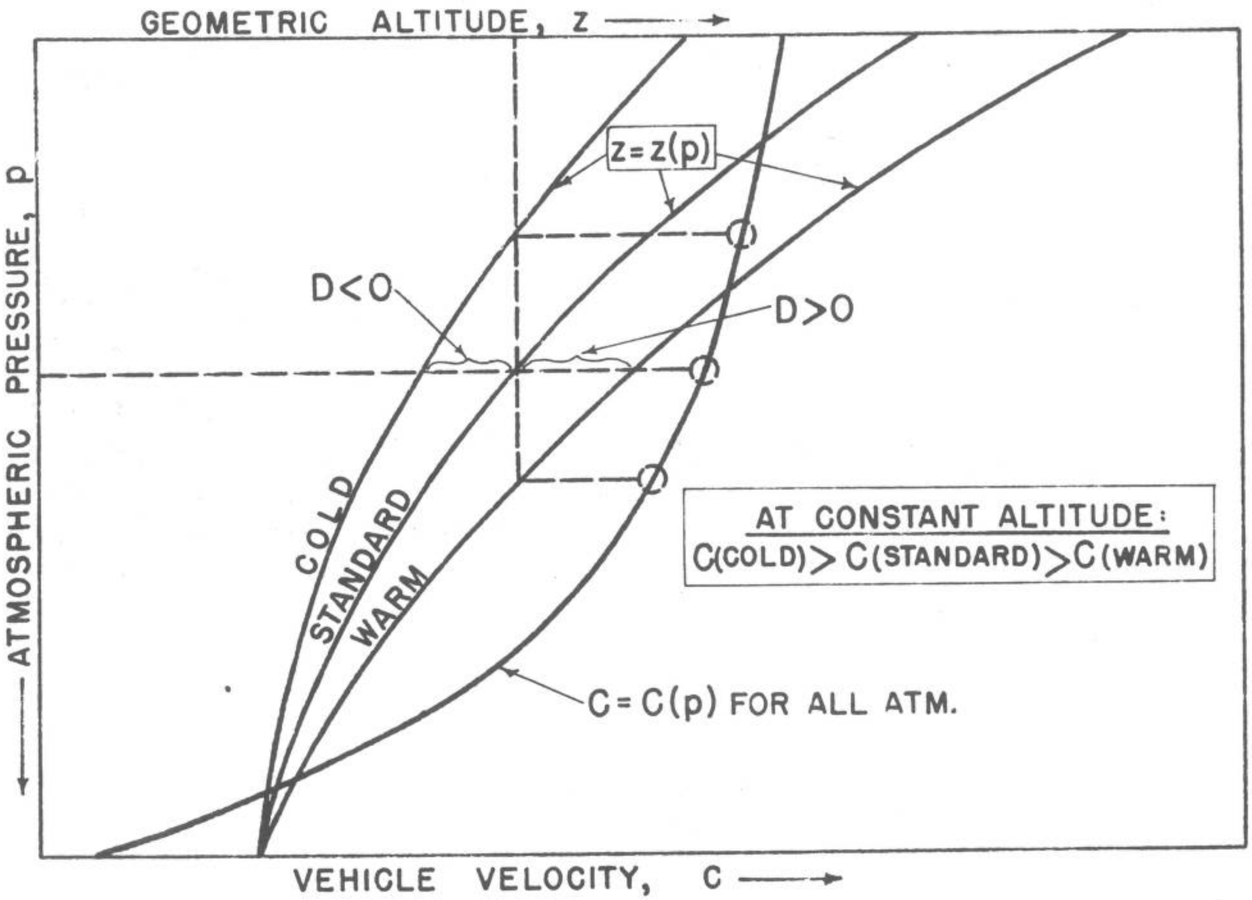

Figure 1. - Illustrating that vehicle velocity is constant at same pressure, but is different at same altitude in a warm and a cold atmosphere.

THE WIND PERTURBATION

The perturbation velocity caused by the wind may be found by several methods and the choice of method depends very much on the required accuracy and on the characteristics of the standard trajectory. If the latter is only slightly curved, the computations are easier and good analytic solutions can be found.

In general, integration of (11) leads to an inhomogeneous linear integral equation of the Volterra type (see, for instance, [7]).

$$\mathfrak{C}_V = -\int_0^{\omega} \mathfrak{A}\cdot(\mathfrak{C}_V - \mathfrak{v})\,d\xi \tag{25}$$

The equation is of the second kind. It may be solved by an iterative procedure analogous to the Liouville-Neumann series used in solving certain scalar integral equations. It is seen that the series

$$\mathfrak{C}_V = \sum_{m=0}^{\infty} (-1)^m \mathfrak{C}_m$$

where

$$\begin{aligned} \mathfrak{C}_0 &= \int_0^{\omega} \mathfrak{A}\cdot\mathfrak{v}\,d\xi \\ \mathfrak{C}_1 &= \int_0^{\omega} \mathfrak{A}\cdot\mathfrak{C}_0\,d\xi \\ &\vdots \\ \mathfrak{C}_m &= \int_0^{\omega} \mathfrak{A}\cdot\mathfrak{C}_{m-1}\,d\xi \end{aligned} \tag{26}$$

satisfies equation (25). The series converges uniformly.

We may also solve equation (11) as a marching problem by means of the finite difference equation

$$\Delta\mathfrak{C} = -\mathfrak{A}\cdot(\mathfrak{C}-\mathfrak{v})\Delta\omega \tag{27}$$

Let $0, \omega_1, \ldots \omega_m$ be values of the independent variable ω at successive points and let $\Delta\omega_i$ be the increment between successive values ω_i and ω_{i+1}. Let further $\mathfrak{A}_i$ and $\mathfrak{v}_i$ denote values at the midpoint and $\mathfrak{C}_{i+1}$ at the end point of this interval. Intervals are chosen so that $\mathfrak{v}_i$ is a data point routinely available from meteorological sources.

Iterated use of (27) leads to the solution

$$\mathbb{c}_m = \sum_{s=0}^{m-1} \prod_{q=s+1}^{m-1} \left(1 - \Delta\omega_q \mathcal{A}_q\cdot\right) \Delta\omega_s \mathcal{A}_s\cdot \mathbb{v}_s \tag{28}$$

The tensor $\mathcal{A}$, which depends on the geometry of the standard trajectory, may be regarded as a constant in slightly curved trajectories, and for this case (28) may be expressed by more simple formulae, as is shown later.

In a right-handed orthogonal system with unit vectors $\mathbb{i}$, $\mathbb{j}$, and $\mathbb{K}$ ($\mathbb{K}$ is vertical upward, $\mathbb{i}$ horizontal in plane of trajectory), $\mathcal{A}$ has the components

$$\mathcal{A} \equiv \begin{Bmatrix} 1+\cos^2\psi, & 0 & , -\sin\psi\cos\psi \\ 0, & 1 & , \quad 0 \\ -\sin\psi\cos\psi & , 0 & , 1+\sin^2\psi \end{Bmatrix} \tag{29}$$

In (28) the perturbation velocity is written as a linear sum of the layer winds above the point in question, the wind in each layer being "weighted" by an appropriate factor. These factors are identical with the empirical influence coefficients referred to briefly in the Introduction. They have here been given a physical interpretation and can be computed from the geometry of the standard trajectory, the re-entry vehicle characteristics, and the pressure difference across the layer. The problem which was outlined in the Introduction has, in principle, been solved.

From the perturbation velocities one arrives at the displacements:

$$\mathbb{r} = \sum \mathbb{c}_i \Delta t_i$$

where the time interval Δt_i spent in the i-th layer is taken from the standard trajectory.

Slightly Curved Trajectories

In many applications the trajectories remain practically straight to the end. In such cases, the wind effect may be computed analytically. In equation (11):

$$\frac{d\mathbb{c}_v}{d\omega} + \mathcal{A}\cdot\mathbb{c}_v = \mathcal{A}\cdot\mathbb{v}$$

the tensor $\mathcal{A}$ is now regarded as a constant independent of ω .

We will apply a Laplace transform to equation (11), indicating the transformation by a bar as follows:

$$\overline{f(r)} = \int_0^\infty e^{-r\omega} f(\omega)\, d\omega$$

With the boundary condition $\mathfrak{C}_V = 0$ when $\omega = 0$ the result is

$$\overline{\mathfrak{C}}_V = \left[1 - r\left(r + \mathcal{A}\right)^{-1}\cdot\right]\overline{\mathcal{V}} \tag{30}$$

where $(r + \mathcal{A})^{-1}$ indicates the reciprocal of the tensor $r + \mathcal{A}$.

The component equations of (30) in the i, j, k-system are, when α , β , and γ are components of $\mathfrak{C}_V$:

$$\bar{\alpha} = \frac{2+(1+\cos^2\psi)r}{(r+1)(r+2)}\,\bar{u}$$

$$\bar{\beta} = \frac{1}{r+1}\,\bar{v} \tag{31}$$

$$\bar{\gamma} = -\frac{\sin\psi\cos\psi}{(r+1)(r+2)}\,\bar{u}$$

where u and v are the i and j components of the wind. There is assumed to be no vertical motion of the air.

The inversion theorem for the Laplace transform gives the solutions:

$$\alpha = \sin^2\psi\int_0^\omega\left(e^{-(\omega-\xi)} + 2\cot^2\psi\, e^{-2(\omega-\xi)}\right)u\, d\xi$$

$$\beta = \int_0^\omega e^{-(\omega-\xi)}\, v\, d\xi \tag{32}$$

$$\gamma = -\sin\psi\cos\psi\int_0^\omega\left(2e^{-2(\omega-\xi)} - e^{-(\omega-\xi)}\right)u\, d\xi$$

Again, if we write the perturbation velocities in the form assumed in the method of linear influence coefficients:

$$\alpha = \sum a_i u_i \tag{33}$$

we find that the influence coefficient a_i can be given the interpretation

$$a_i = \left(\sin^2\psi\, e^{-(\omega-\omega_i)} + 2\cos^2\psi\, e^{-2(\omega-\omega_i)}\right)\Delta\omega_i \qquad (34)$$

with similar expressions for the remaining components.

Uniform Wind

The effect of a uniform wind is readily obtained:

$$\begin{aligned} \alpha &= (1-\sin^2\psi e^{-\omega} - \cos^2\psi e^{-2\omega})u \\ \beta &= (1-e^{-\omega})v \\ \gamma &= \sin\psi\cos\psi(e^{-\omega}-e^{-2\omega})u \end{aligned} \qquad (35)$$

We notice that a vertical perturbation velocity arises even if we have assumed that the air motion is strictly horizontal. The effect is caused by the "weathercocking" of the vehicle. As it turns into the wind, a positive range component of the wind will cause a downward-directed perturbation velocity. It is hereby brought to our attention that the effect of a wind is not merely to displace the vehicle along and across the range, but also to accelerate or decelerate its earthward motion. In other words, the wind as well as the density should, in principle, be considered in altimetry problems where our concern is the distance of events from the ground.

A numerical example shows that the maximum vertical displacement of a re-entry vehicle of ballistic coefficient 2,000 lb. $^{-1}$sq. (sq. ft)$^{-1}$ re-entry angle 20 deg., and re-entry velocity 22,000 ft. sec.$^{-1}$, amounts to 86 ft. for a uniform range component of the wind of 50 kt. The wind is assumed to blow uniformly all the way from 100,000 ft., to the surface. This example probably represents an extreme case of vertical displacement due to wind.

To estimate the displacement from the perturbation velocity, it will, in most cases be sufficiently accurate to use an approximate solution for the re-entry speed. For instance, the solution first suggested by Allen and Eggers [1] may be used

$$C = C_E e^{-\omega}$$

where C_E is the value at the "top" of the atmosphere.

In view of (4), (9), (35), and (24), we obtain for the displacements (in the case of a uniform wind)

$$\Delta X = \int_0^\omega \alpha\, dt = \frac{R u}{g \sin\psi C_E} \int_0^\omega \Theta \frac{(e^{\xi}-1) + \cos^2\psi(1-e^{-\xi})}{\xi}\, d\xi$$

$$\Delta y = \int_0^\omega \beta\, dt = \frac{R v}{g \sin\psi C_E} \int_0^\omega \Theta \frac{1-e^{-\xi}}{\xi}\, d\xi \qquad (36)$$

$$\Delta Z = \int_0^\omega \gamma\, dt = \frac{R \cos\psi\, w}{g C_E} \int_0^\omega \Theta \frac{1-e^{-\xi}}{\xi}\, d\xi$$

In (36) the absolute temperature Θ and the gas constant R have been inserted from the equation of state:

$$P = \frac{p}{R\Theta} \qquad (37)$$

The integrals in (36) can be evaluated from tables related to the exponential integral. (See, for instance, [2].)

Figure 2 shows the maximum vertical displacement in feet as a function of the weight-to-drag ratio and the trajectory angle for a uniform range wind of 10 m. sec.$^{-1}$, based on a re-entry speed of 22,000 ft. sec.$^{-1}$ and the U. S. Standard Atmosphere, 1962 [8].

Figure 3 shows the corresponding range displacement near the ground.

INERTIAL ALTIMETRY PROBLEMS

Because of the property of the vehicle velocity of being independent of density, when viewed as a function of pressure, the meteorological influences on inertial altimetry systems can readily be calculated. It is assumed that the basic trajectory for a given set of vehicle characteristics (mg/C_DA) and re-entry conditions (ψ and C_E) has been obtained, either by a numerical integration or by using some approximate formula such as (24). In other words, we have at our disposal in analytical or tabular form the basic relationship given in (22).

In the following sections certain analyses are made to determine the altitude errors ΔZ associated with the most typical inertial events used in altimetry systems. Wind, while having a slight effect on vertical position, can be ignored for most systems as pointed out in section on Uniform Wind.

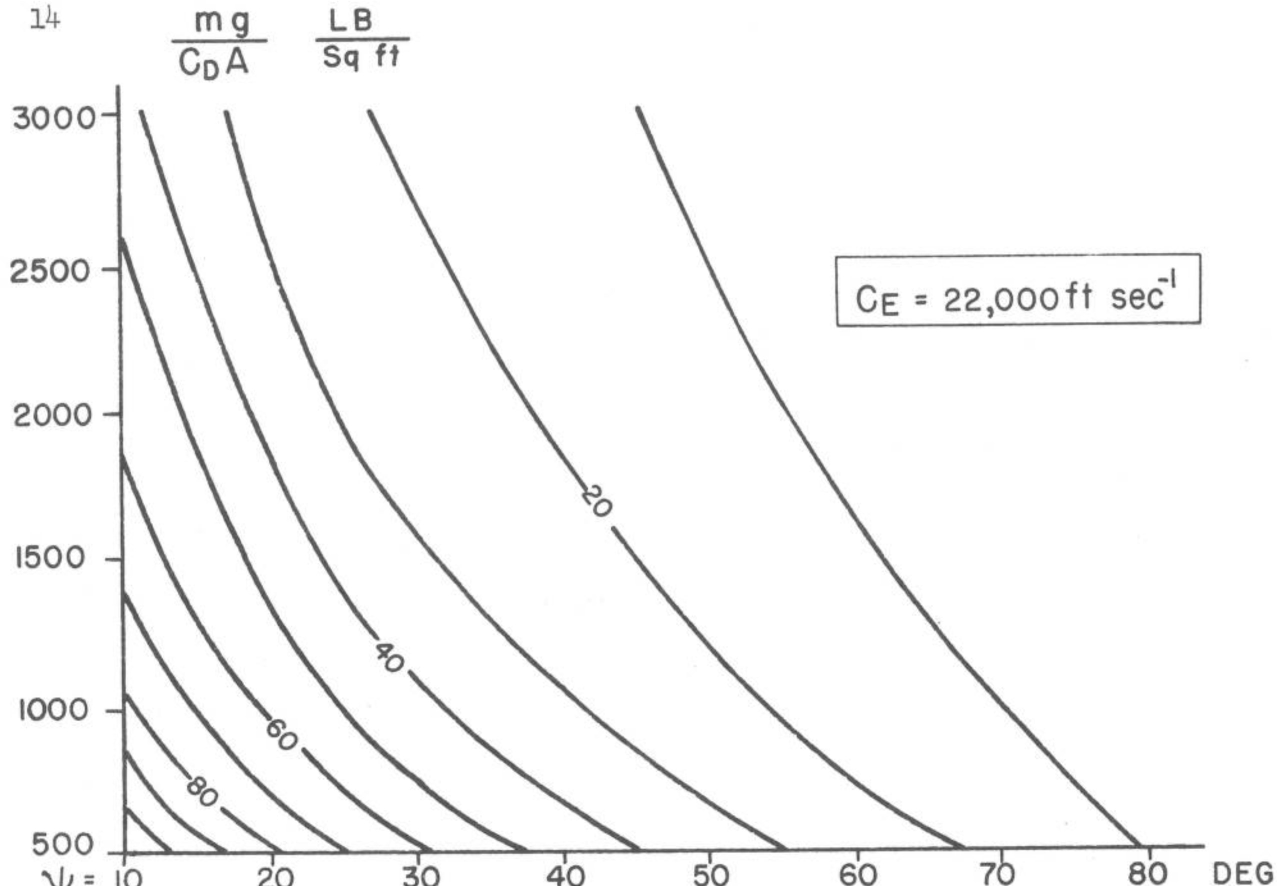

Figure 2. - Vertical displacement (ft.) at 1000 mb. due to a uniform range-wind of 10 m. sec.$^{-1}$.

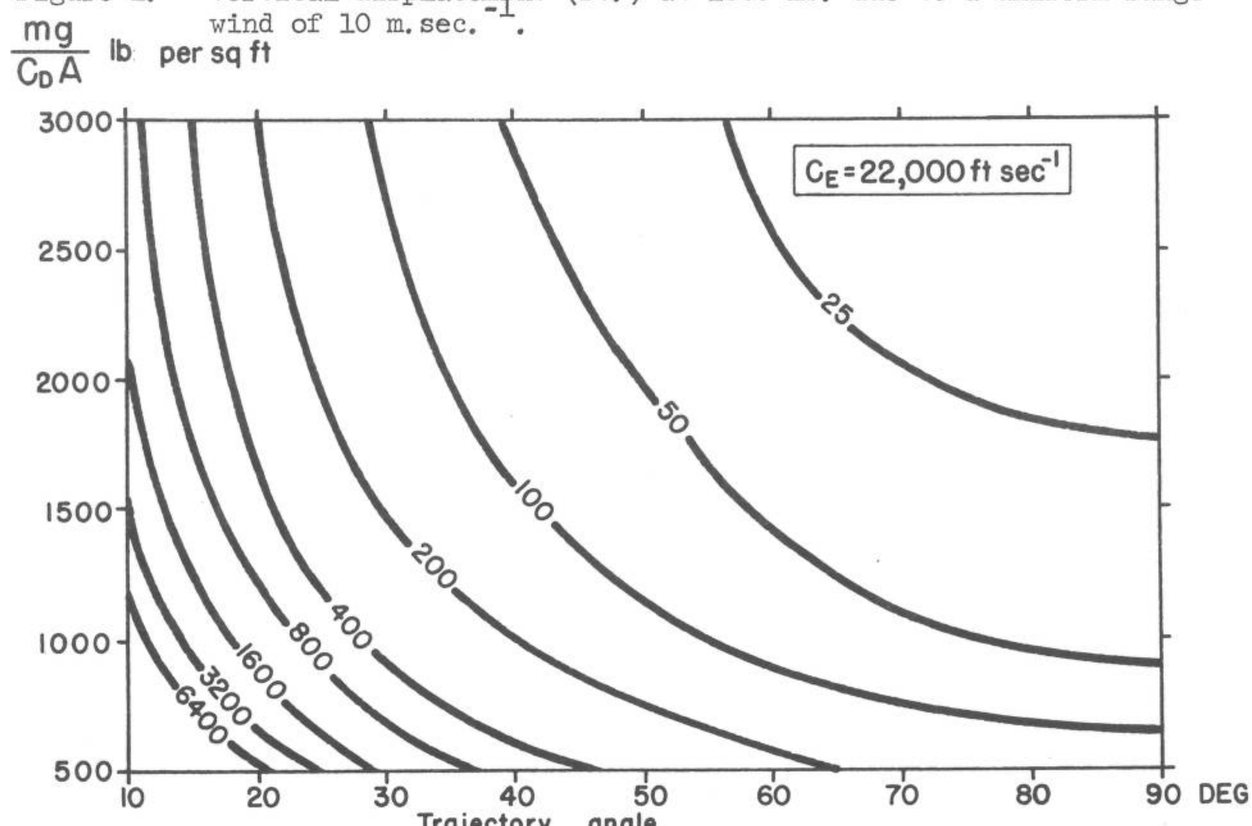

Figure 3. - Range displacement (ft.) at 1000 mb. caused by a uniform range-wind of 10 m. sec.$^{-1}$ throughout the atmosphere. Re-entry speed 22,000 ft. sec.$^{-1}$.

Practical applications of the formulae in this paragraph may be found in references [3, 4, 5, and 6].

Discrete Deceleration Signal. The drag deceleration, measured in g-units, is determined by

$$N = \frac{p}{R(\Theta+\theta)} \frac{C_D A}{2mg} C^2 \tag{38}$$

As C is a unique function of p, N in (38) appears as a function of pressure and temperature only, and N = constant defines a curve of constant deceleration in a diagram which has pressure and temperature as axes, as shown in figure 4.

A temperature anomaly θ at the programed pressure p_N of the deceleration event, will cause a pressure error, Δp_N, which can be determined by varying equation (38). We find

$$\Delta p_N = \frac{C\theta p_N}{\Theta\left(C+2p_N\frac{dC}{dp}\right)} \tag{39}$$

All parameters in (39) are evaluated at pressure p_N.

For the altitude error we obtain:

$$\Delta Z_N = -\frac{RC\theta}{g\left(C+2p_N\frac{dC}{dp}\right)} + D(p_N) \tag{40}$$

where $D(p_N)$ is the D -value at pressure p_N

Signal from Integrating Accelerometer. The integral of the drag deceleration over time is, in units of g sec:

$$I = \int_0^p \frac{C_D A}{2mg}(P+\rho)C^2\,dt = \int_0^p \frac{C_D A}{2mg^2\sin\psi}\,C\,dp = I(p) \tag{41}$$

We have arrived at the important result that the output of the integrating accelerometer is independent of the density profile and, like the vehicle velocity, is a unique function of pressure. Hence, the pressure error of the integrating accelerometer due to density departure from standard is zero:

$$\Delta p_I = 0$$

and the altitude error is:
$$\Delta Z_I = D(p_I)$$

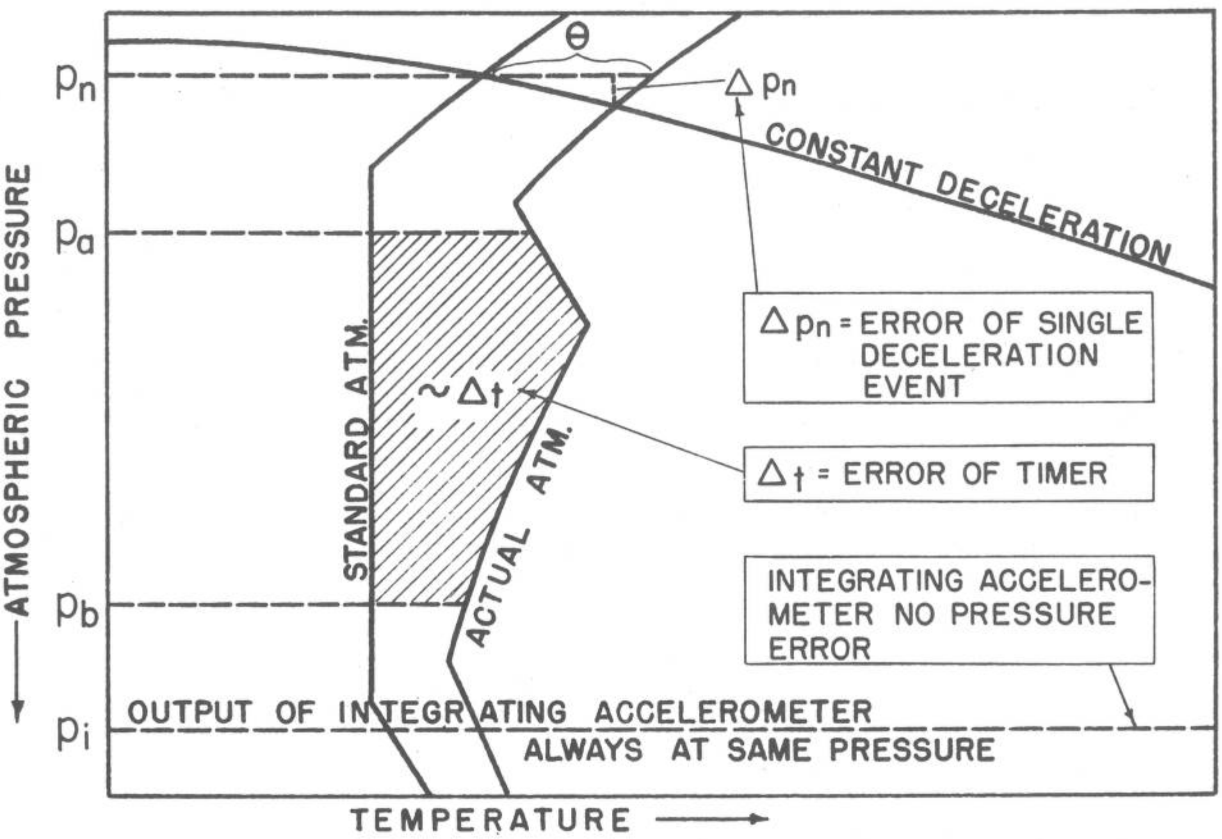

Figure 4. - Illustrating effects of non-standard atmosphere on inertial altimetry components.

(42)

where p_I is the pressure at which the output from the instrument is programed to reach the pre-determined value I.

Time of Flight. Sometimes, timing devices form components of the altimetry systems, and the question of the time of flight across a pressure layer arises. The time t taken to cross from pressure p_a to p_b is given by

$$t = \int_{p_a}^{p_b} dt = -\int_{p_a}^{p_b} \frac{dz}{\sin\psi C} = \frac{R}{g}\int_{p_a}^{p_b} \frac{\Theta + \theta}{C\,p \sin\psi}\, dp$$

and the time increment Δt due to the temperature anomaly is:

$$\Delta t = \frac{R}{g} \int_{pa}^{pb} \frac{\theta \, dp}{C_p \sin\psi} \tag{43}$$

In evaluating (43) it is sometimes favorable to form the function $\xi = \xi(p)$ defined by:

$$d\xi = \frac{C_E}{C \sin\psi} \frac{dp}{p}$$

For example, formula (24) for C leads to the function:

$$\xi = Ei(\omega) = \int_{-\infty}^{\omega} e^{x} \frac{dx}{x}$$

$Ei(\omega)$ is the exponential integral.

Equation (43) then takes the simple form:

$$\Delta t = \frac{R}{g C_E} \int_{pa}^{pb} \theta \, d\xi$$

If the pressure axis in figure 4 is scaled by the function ξ, the hatched area is a direct measure of Δt and the integral in (43) can be evaluated by the sum:

$$\Delta t = \frac{R}{g C_E} \sum \theta_i \Delta \xi_i$$

Error Distributions. Analysis of the meteorological errors of inertial altimetry systems according to the procedures outlined above usually leads to expressions of the form

$$\Delta Z = \sum_i A_i(p_i)\,\theta + D(p_T) \tag{44}$$

The summation term may result from the fact that integrals such as (43) have to be evaluated by summation. The summation occurs over fairly large intervals, as meteorological data are stored and processed statistically at only a limited number of selected pressures. Typically, such pressures are:

1,000, 850, 700, 500, 300, 250, 200, 150, and 100 millibars. The functions $A_i(p_i)$, here written as functions of pressure, usually reduce to constants in a definitive altimetry program. Sometimes, though, the optimum program is being sought and now the p_i's may be regarded as variables. The object is now to find the set of p_i's which gives the minimum dispersion of ΔZ In such optimization problems other error sources than the meteorological ones may have to be considered during the optimization process, as they as well may be pressure dependent. The contribution of the non-meteorological error sources (for instance, timing device, accelerometer, activating mechanism, etc.) may often be written as

$$\sum_q B_q(p_q)\,\varepsilon_q$$

where the functions $B_q(p_q)$ indicate the pressure dependence and ε_q are the non-meteorological errors.

The total system error is then:

$$\Delta Z = \sum_i A_i(p_i)\,\theta_i + D(p_T) + \sum_q B_q(p_q)\,\varepsilon_q \tag{45}$$

Note from Eq. (45) that only one D-value is involved. This is usually the D-value at the terminal point in the program, at a programed pressure p_T

In seeking the statistical distribution of ΔZ during a certain month and for a certain location, the assumption is usually made that the meteorological variables θ and D are normally distributed. This is a fair assumption up to about 2 standard deviations and is satisfactory for the bulk of the distribution, but should be used cautiously for estimating the probability of extreme values.

The non-meteorological and the meteorological errors and also the non-meteorological errors among themselves are assumed to be un-correlated. There is usually no reason to assume otherwise. However, the meteorological terms have highly significant correlations.

As an example, figure 5 shows the correlation of the simultaneous temperature at two points in the same vertical.

Under these assumptions, we obtain for the standard deviation of ΔZ , using the symbol $\sigma(\phi)$ to denote the standard deviation of ϕ :

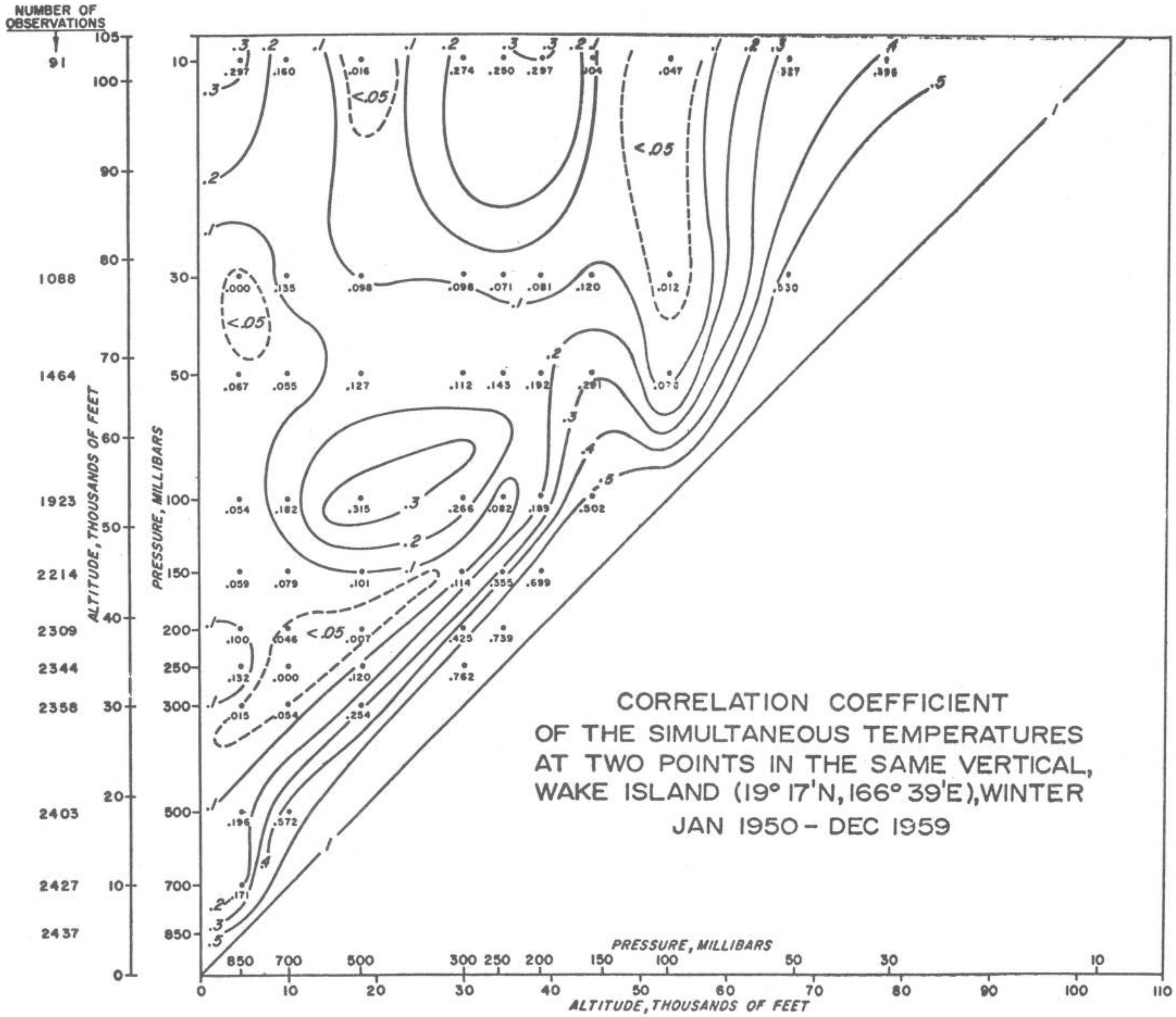

Figure 5. - Temperature correlations.

$$
\begin{aligned}
\sigma^2(\Delta z) &= \sigma^2(D) + \sum_q B_q^2(p_q)\,\sigma^2(\varepsilon_q) \\
&+ \sum_i \sum_j \left(A_i(p_i)\,A_j(p_j)\,r(\theta_i,\theta_j)\,\sigma(\theta_i)\,\sigma(\theta_j)\right) \\
&+ \sum_i \left(A_i(p_i)\,r(D,\theta_i)\,\sigma(D)\,\sigma(\theta_i)\right)
\end{aligned}
\tag{46}
$$

Here, $r(x,y)$ indicates the linear correlation coefficient between x and y.

The statistical quantities generally needed for the estimate of the error distribution of Δz are seen to be:

(1) Standard deviation of temperature at various pressures.

(2) Standard deviation of D-values at various pressures.

(3) Matrices of the correlation coefficient of temperatures at two points in the vertical.

(4) Matrices of the correlation coefficient of the D-value at a given pressure and the simultaneous temperature at various points above this pressure.

This information is available in the meteorological archives for many points on the globe. The chief source of this type of data is probably the Environmental Technical Application Center, Air Weather Service, USAF.

As an interesting sidelight, one might speculate over what a system with the least meteorological dispersion would look like. In principle, and considering only meteorological errors, the minimal dispersion of any inertial system is equal to that of the D-value at the lowest point in the trajectory. This refers to a system which has no timer but has an integrating accelerometer which uses the terminal output as a signal. The practicability of such a system has not been considered.

Figure 6 shows the standard deviation of D close to the surface during the month of January. Values range from as high as 500 ft. in middle latitudes to less than 50 ft. in the Tropics.

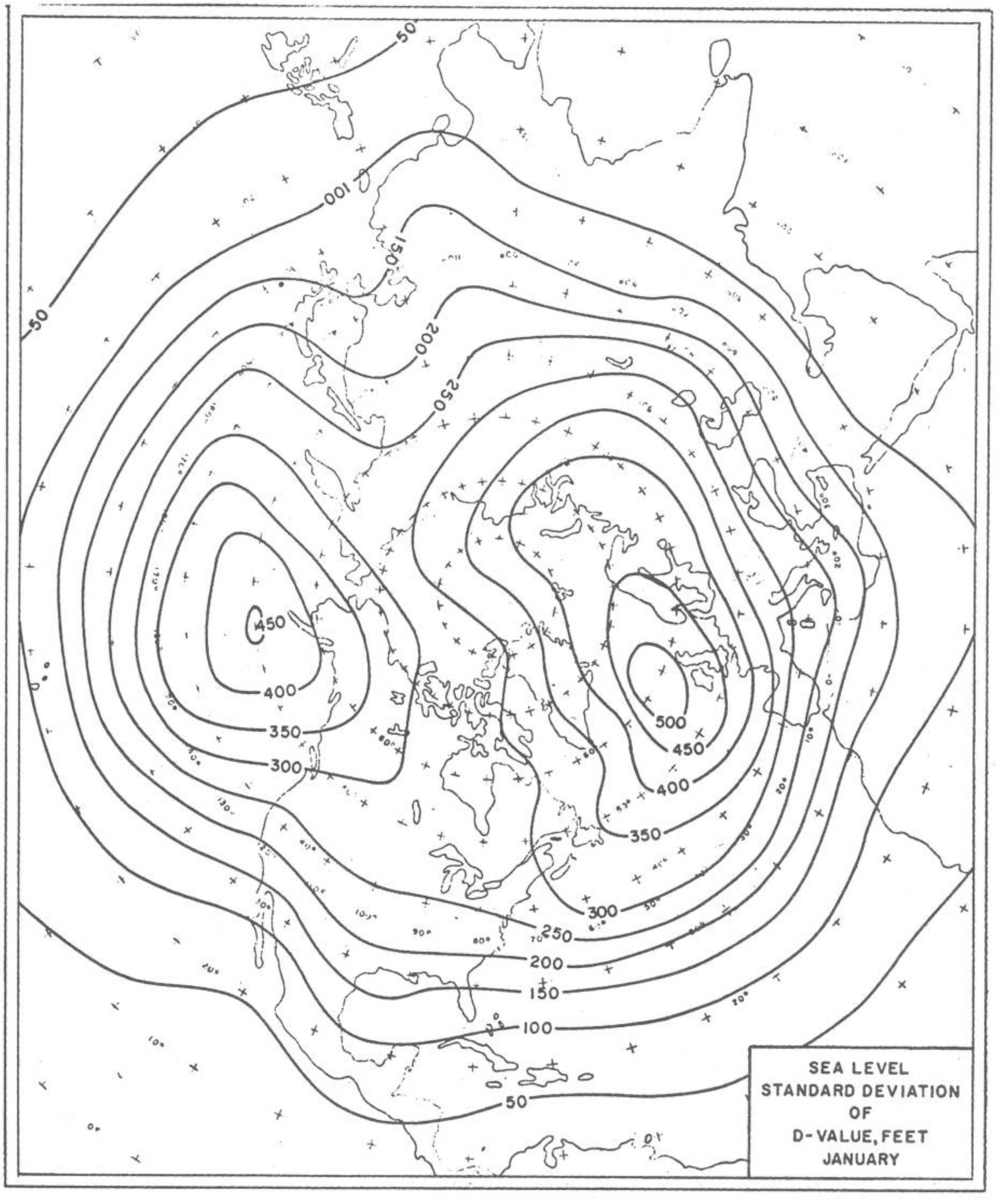

Figure 6. - D-value, Northern Hemisphere, January.

SYMBOLS

$\mathbb{C}$ = Velocity relative to the earth under standard conditions, i.e., standard density and no wind, scalar value C , re-entry value C_E

$\mathfrak{c}$ = Perturbation velocity caused by winds and density anomalies. $\mathbb{C} + \mathfrak{c}$ is velocity in real atmosphere, $\mathfrak{c} = \mathfrak{c}_\rho + \mathfrak{c}_V$

$\mathfrak{c}_\rho$ = Perturbation velocity due to density anomalies.

$\mathfrak{c}_V$ = Perturbation velocity due to wind.

$\mathbb{V}$ = Wind.

g = Acceleration of gravity.

Ω = Rotation of the earth.

$\mathfrak{T}$ = Unit vector tangent to trajectory.

P = Density of standard atmosphere.

ρ = Departure of density from standard.

p = Atmospheric pressure.

Z = Geometric altitude.

C_D = Drag coefficient.

A = Cross section of re-entry vehicle.

m = Mass of re-entry vehicle.

s = Distance along trajectory.

t = Time.

ψ = Trajectory angle.

ω = Non-dimensional pressure variable.
$\frac{C_D A}{2mg \sin\psi} p$

N = Drag deceleration, in g-units.

$\mathcal{A}\cdot$ = Dyadic operator = $(1 + \mathbb{T}\mathbb{T}\cdot)$

Θ = Temperature, °K., of standard atmosphere.

θ = Departure of virtual temperature from standard. (°C.)

R = Gas constant for dry air.

$D(p)$ = D-value, i.e., altitude of pressure p in actual atmosphere minus altitude of same pressure in standard atmosphere.

α, β, γ = Components of perturbation velocity along $\mathbb{i}$, $\mathbb{j}$, and $\mathbb{K}$.

$\Delta x, \Delta y, \Delta z$ = Components of vehicle displacement.

u, v = Range and cross-range components of wind.

$\sigma(\phi)$ = Standard deviation of ϕ.

$r(x, y)$ = Correlation coefficient between x and y.

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